Nonconservative Lifted Convex Conditions for Stability of Discrete-Time Switched Systems Under Minimum Dwell-Time Constraint

Weiming Xiang, Hoang-Dung Tran, and Taylor T. Johnson

***Abstract*—In this note, a novel conception called virtual clock, which is defined by an artificial timer over a finite cycle, is introduced for stability analysis of discrete-time switched linear systems under minimum dwell-time constraint. Two necessary and sufficient conditions associated with a virtual clock with a sufficient length are proposed to ensure the global uniform asymptotic stability of discrete-time switched linear systems. For the two nonconservative stability criteria, the lifted version maintains the convexity in system matrices. Based on the lifted convex conditions, the extensions to ℓ_2-gain computation and $\mathcal{H}_\infty$ control problems are presented in the sequel. In particular, a novel virtual-clock-dependent controller is designed, which outperforms the traditional mode-dependent and common gain controllers. Several numerical examples are provided to illustrate our theoretic results.**

***Index Terms*—Dwell time, $\mathcal{H}_\infty$ control, ℓ_2-gain, stability, switched system.**

I. Introduction

A switched system is composed of a finite number of dynamic subsystems described by differential or difference equations, along with a switching rule governing the switching among them. The motivation for studying switched systems comes from the fact that switched systems can be efficiently used to model many practical systems that are inherently multimodal. In this regard, several dynamic subsystem models are required to describe system behaviors. Among the large variety of problems studied in theory and encountered in practice, stability analysis is one of the core problems in the field of switched systems, which has attracted considerable research attention over the last decade; See [1]–[3] and the references cited therein. One way to study the stability of switched systems is through the notions of dwell time and average dwell time, which are used to characterize the switching rate of a switched system. In the framework of Lyapunov function, a number of methods have been reported and demonstrated to be effective in stability analysis [4]–[11]. Reachability analysis methods have been utilized to solve the stability analysis problems in [12]–[17].

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Recently, a promising method called the *lifting* approach has been proposed in both continuous-time [18], [19] and discrete-time [20], [21] domains. Those lifted conditions are equivalent convexifications of the well-known results in [6], [7], and [22]. This convex feature can significantly facilitate further extensions from stability results to other relevant problems, such as robust stability analysis and input–output performance analysis. Some recent generalizations and applications of this lifted idea can be found in terms of more general homogeneous Lyapunov functions [23], extreme cases such as the switched system being fully composed of unstable subsystems [24], hybrid positive systems [25], [26], stochastic systems [27], [28], uncertain systems [29], periodic systems [30], and fault-tolerant control [31]. In the framework of quadratic Lyapunov functions, those proposed lifted conditions expressed in terms of linear matrix inequalities (LMIs) may sometimes provide tight results; however, they are not necessary in general. Thus, the main problem addressed in this paper arises:

1) Can the necessity for the stability of switched systems be also recovered by further generalizing the lifting approach? Namely, can we derive a nonconservative stability criterion for switched systems in the framework of lifting approach?

In [32], a nonconservative stability result is derived in the framework of polynomial functions, where the nonconservativeness can be achieved with a sufficiently high degree of the polynomial function. In [33]–[35], the lifting approach is used for the stability analysis of switched systems under constrained switching signals and controller design [36], where the switching constraints are mostly characterized by switching graphs or language. In this note, we will answer this question for a class of time-dependent switched systems under minimum dwell-time constraint by generalizing the results in [20] and [21]. To make the generalization and inspired by [33]–[36], we introduce a novel conception called *virtual clock* for discrete-time switched linear systems, which generalizes the framework of dwell time and plays a fundamental role for achieving nonconservativeness in stability analysis. It also needs to be noted that we further explore the relationship between lifting approach and the well-known method in [7]. With the help of a virtual clock, two necessary and sufficient conditions for global uniform asymptotic stability of discrete-time switched linear systems under minimum dwell-time constraint are proposed. It is worthwhile mentioning that the nonconservative lifted convex version can be viewed as an extension of the sufficient lifted convex condition in [20], and the other one can be also viewed as a nonconservative extension of the well-known result in [7]. By the merit of the convexity of lifted conditions, the stability result can be extended to solve the ℓ_2-gain computation problem. Based on the virtual clock, a class of $\mathcal{H}_\infty$ virtual-clock-dependent feedback controllers are designed, which have a better performance than conventional mode-dependent and common gain controllers.

Notations: $\mathbb{N}$ represents the set of natural numbers, $\mathbb{R}$ denotes the field of real numbers, $\mathbb{R}^+$ is the set of nonnegative real numbers, and $\mathbb{R}^n$ stands for the vector space of all n-tuples of real numbers, and $\mathbb{R}^{n\times n}$ is the space of $n \times n$ matrices with real entries. The set of $n \times n$

(positive definite) symmetric matrices is denoted by ($\mathbb{S}^n_{\succ 0}$) $\mathbb{S}^n$. $\|\cdot\|$ stands for Euclidean norm. For a set $\mathcal{A}$, $|\mathcal{A}|$ denotes its cardinality. The notation $A \succ 0$ means A is real symmetric and positive definite. $A \succ B$ means that $A - B \succ 0$. $A^\top$ denotes the transpose of A. In addition, in symmetric block matrices, we use * as an ellipsis for the terms that are induced by symmetry. A continuous function $\alpha : \mathbb{R}^+ \to \mathbb{R}^+$ is a class $\mathcal{K}$ function if it is strictly increasing and $\alpha(0) = 0$. Moreover, a function $\beta : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ is a class $\mathcal{KL}$ function if, for each fixed s, the function $\beta(r, s)$ is a class $\mathcal{K}$ function with respect to r and, for each fixed r, the function $\beta(r, s)$ decreases with respect to s and $\beta(r, s) \to 0$ as $s \to \infty$. $\mathrm{int}[x]$ denotes the integer part of x. For two integers k_1 and k_2, $k_1 \le k_2$, we define $\mathcal{I}[k_1, k_2] \triangleq \{k_1, k_1 + 1, \ldots, k_2\}$.

II. Preliminaries and Problem Formulation

In this note, we consider a class of linear switched systems in the following form:

$$x(k+1) = A_{\sigma(k)} x(k)$$
$$x(0) = x_0 \tag{1}$$

where $x(k), x_0 \in \mathbb{R}^n$ are the system state vector and initial state, respectively. The switching signal σ is defined as $\sigma : \mathbb{N} \to \mathcal{I}[1, N]$, where N is the number of subsystems involved in the switched system. Let $k_0 = 0$ be the initial time by convention, and let switching instants be denoted by k_ℓ, $\ell \in \mathbb{N}\backslash\{0\}$. In addition, we denote $\mathscr{S} \triangleq \{k_\ell\}_{\ell \in \mathbb{N}}$. Calling $\mathscr{D}_\tau$ as the set of all switching policies with dwell time τ, that is, the set of all $\sigma(k)$ for which the time intervals between any successive switching instants of $\sigma(k)$ satisfy $k_{\ell+1} - k_\ell \ge \tau$, $\ell \in \mathbb{N}\backslash\{0\}$.

Definition 1: [37] The equilibrium $x = 0$ of system (1) is globally uniformly asymptotically stable (GUAS) under switching signal $\sigma(k)$ if, for any initial condition $x(0)$, there exists a class $\mathcal{KL}$ function β such that the solution of system (1) satisfies $\|x(k)\| \le \beta(\|x(0)\|, k)$ $\forall k \in \mathbb{N}$.

One of the basic problems for the stability analysis of switched systems is to determine the minimum dwell time guaranteeing the GUAS, known as the *minimum dwell-time problem* defined as follows:

$$\tau_{\min} = \min\{\tau > 0 : \text{System (1) } is \text{ GUAS} \forall \sigma(k) \in \mathscr{D}_\tau\}. \tag{2}$$

For the minimum dwell-time problem, two well-known results are recalled.

Theorem 1: [7] Given that for some positive scalar τ, if there exist a collection of symmetric matrices $P_i \in \mathbb{S}^n_{\succ 0}$, $i \in \mathcal{I}[1, N]$, such that the following conditions:

$$A_i^\top P_i A_i - P_i \prec 0 \tag{3}$$
$$(A_i^\tau)^\top P_j A_i^\tau - P_i \prec 0 \tag{4}$$

hold for all $i, j \in \mathcal{I}[1, N]$, $i \ne j$, then system (1) is GUAS with any switching signal $\sigma(k) \in \mathscr{D}_\tau$.

In terms of Lyapunov functions with quadratic structure, the abovementioned result seems to be the best possible so far. However, the LMIs in (4) depend on the exponential term A_i^τ, which is not convex in system matrices A_i. This underlying nonconvexity prevents further extensions. To circumvent this, a set of alternative equivalent conditions in terms of LMIs that are affine in the system matrices have been proposed in [20], which are called lifted convex conditions.

Theorem 2: [20] Given that for some positive scalar τ, if there exist matrix sequences $P_i : \mathcal{I}[0, \tau] \to \mathbb{S}^n_{\succ 0}$, $i \in \mathcal{I}[1, N]$, such that the following conditions:

$$A_i^\top P_i(\tau) A_i - P_i(\tau) \prec 0 \tag{5}$$
$$A_i^\top P_i(k+1) A_i - P_i(k) \prec 0 \tag{6}$$
$$P_i(0) - P_j(\tau) \prec 0 \tag{7}$$

hold for all $i, j \in \mathcal{I}[1, N]$, $i \ne j$ and $k \in \mathcal{I}[0, \tau - 1]$, then system (1) is GUAS with any switching signal $\sigma(k) \in \mathscr{D}_\tau$.

It has been demonstrated that the lifted condition in Theorem 2 is an equivalent convexification of Theorem 1, at the expense of more computational costs in solving LMIs. By the merit of convexity, Theorem 2 can be easily extended to those problems that Theorem 1 is incapable of dealing with, for instance, robust stability analysis, controller design, and input–output performance analysis. Though Theorem 2 with a set of lifted convex conditions has some obvious advantages over Theorem 1, it is, in essence, a sufficient stability criterion that is the same as Theorem 1. In this note, our main aim is to further develop the lifting idea to derive a nonconservative stability criterion for switched system (1).

III. Necessary and Sufficient Lifted Convex Stability Conditions

A. Virtual Clock and Admissible Cycles

The lifted conditions in [20] are expressed in the form of a sequence of interdependent LMIs defined over a dwell-time interval $[0, \tau]$. In order to further develop the lifting idea, a novel notion we call the virtual clock is first introduced.

Definition 2: A virtual clock is defined by

$$\mathscr{C}_L \triangleq \{\theta(k), [0, L-1]\} \tag{8}$$

where $\theta(k)$ is an artificial timer in the form of

$$\theta(k) = k - L\mathrm{int}[k/L],\ k \in \mathbb{N} \tag{9}$$

taking values in $[0, L-1]$.

Given a dwell-time constraint $k_{\ell+1} - k_\ell \ge \tau$ $\forall \ell \in \mathbb{N}\backslash\{0\}$ and a virtual clock $\mathscr{C}_L$ with a length $L \ge \tau$, the conception of an admissible switching path with respect to $\{\mathscr{C}_L, \tau\}$ is introduced.

Definition 3: Given a dwell time τ and a virtual clock $\mathscr{C}_L$, $L \ge \tau$, a switching path $\mathcal{S} \triangleq \{i_0, \ldots, i_{L-1}\}, i_0, \ldots, i_{L-1} \in \mathcal{I}[1, N]$, over cycle $[0, L-1]$, is an admissible path with respect to $\{\mathscr{C}_L, \tau\}$, if all the switchings in path $\mathcal{S}$ satisfy the dwell-time constraint. The index set of all admissible switching paths is denoted by $\mathscr{A}(L, \tau)$.

Furthermore, the concatenation of two admissible switching paths needs to be considered to fully characterize the evolution of a switching signal. The set of postadmissible paths with respect to an admissible switching path is defined as follows.

Definition 4: Given two admissible switching paths $\mathcal{S}_1 \triangleq \{i_0, \ldots, i_{L-1}\}$ and $\mathcal{S}_2 \triangleq \{j_0, \ldots, j_{L-1}\}$ with respect to $\{\mathscr{C}_L, \tau\}$, $\mathcal{S}_2$ is the postadmissible switching path of $\mathcal{S}_1$, if all the switchings in the concatenation path $\{\mathcal{S}_1, \mathcal{S}_2\} \triangleq \{i_0, \ldots, i_{L-1}, j_0, \ldots, j_{L-1}\}$ satisfy the dwell-time constraint. The index set of all postadmissible switching paths of switching path $i \in \mathscr{A}(L, \tau)$ is denoted by $\mathscr{PA}(i)$, $i \in \mathscr{A}(L, \tau)$.

The following example is presented to illustrate the notions of admissible and postadmissible paths.

Example 1: Consider a switched system with two modes; the dwell time is assumed to be $\tau = 2$, and the length of the cycle of the virtual clock is chosen to be $L = 3$. Explicitly, switching paths $\{1, 2, 1\}$ and $\{2, 1, 2\}$ violate the dwell-time constraint since the time between two switchings is 1, which is less than the dwell time 2. By excluding those two inadmissible switching paths, six admissible switching paths remain, and they are denoted by $\{\mathcal{S}_1, \mathcal{S}_2, \mathcal{S}_3, \mathcal{S}_4, \mathcal{S}_5, \mathcal{S}_6\}$, where $\mathcal{S}_1 \triangleq \{1,1,1\}$, $\mathcal{S}_2 \triangleq \{1,1,2\}$, $\mathcal{S}_3 \triangleq \{1,2,2\}$, $\mathcal{S}_4 \triangleq \{2,1,1\}$, $\mathcal{S}_5 \triangleq \{2,2,1\}$, and $\mathcal{S}_6 \triangleq \{2,2,2\}$. For these six admissible switching paths, the index set is denoted by $\mathscr{A}(3,2) = \{1,2,3,4,5,6\}$.

Then, we take $\mathcal{S}_1$, for example, to determine the postadmissible switching paths. By excluding the inadmissible postswitching path $\mathcal{S}_4$ since $\{\mathcal{S}_1, \mathcal{S}_4\} \triangleq \{1,1,1,2,1,1\}$ obviously violates the dwell-time constraint, the index set of postadmissible switching paths of $\mathcal{S}_1$ is $\mathscr{P}\mathscr{A}(1) = \{1,2,3,5,6\}$.

By the definitions of admissible path and postadmissible path, we can see that set $\mathscr{A}(L,\tau)$ includes all of the admissible switching paths with dwell-time restriction, and $\mathscr{P}\mathscr{A}(i)\ \forall i \in \mathscr{A}(L,\tau)$ covers all the admissible concatenations of two successive admissible switching paths. Therefore, it can be concluded that all the evolutions of switching signal $\sigma(k)$ under the dwell-time switching constraint are involved in sets $\mathscr{A}(L,\tau)$ and $\mathscr{P}\mathscr{A}(i)\ \forall i \in \mathscr{A}(L,\tau)$.

B. Nonconservative Stability Criterion

Based on the conception of a virtual clock, the main contribution, that is, the two necessary and sufficient conditions for the stability of discrete-time switched linear system (1), is presented in the following theorem.

Theorem 3: Consider the switched system (1). The following three statements are equivalent.

a) Switched system (1) is GUAS with any switching signal $\sigma(k) \in \mathscr{D}_\tau$.

b) There exist a scalar $L \geq \tau$ and symmetric matrix sequences $P_i : \mathcal{I}[0,L] \to \mathbb{S}^n_{\succ 0}$, $i \in \mathscr{A}(L,\tau)$ such that

$$A_{i_k}^\top P_i(k+1) A_{i_k} - P_i(k) \prec 0,\ i_k \in \mathcal{I}[1,N]$$
$$i \in \mathscr{A}(L,\tau),\ k = 0,\ldots,L-1 \tag{10}$$
$$P_i(0) - P_j(L) \prec 0,\ i \in \mathscr{P}\mathscr{A}(j),\ j \in \mathscr{A}(L,\tau). \tag{11}$$

c) There exist a scalar $L \geq \tau$ and symmetric matrices $P_i \in \mathbb{S}^n_{\succ 0}$, $i \in \mathscr{A}(L,\tau)$ such that

$$\left(\prod_{h=1}^{L} A_{i_{L-h}}\right)^\top P_i \left(\prod_{h=1}^{L} A_{i_{L-h}}\right) - P_j \prec 0$$
$$i \in \mathscr{P}\mathscr{A}(i),\ j \in \mathscr{A}(L,\tau) \tag{12}$$

where $\prod_{h=1}^{L} A_{i_{L-h}} = A_{i_{L-1}} \ldots A_{i_0}$.

Proof: (*a*) $\Rightarrow$ (*b*): Given any $P_i(L) \in \mathbb{S}^n_{\succ 0}$, $i \in \mathscr{A}(L,\tau)$ and any $X_i(k) \in \mathbb{S}^n_{\succ 0}$, $k \in \mathcal{I}[0,L-1]$, $i \in \mathscr{A}(L,\tau)$, we can define $P_i(k)$ in the form of

$$P_i(k) = \left(\prod_{h=1}^{L-k} A_{i_{L-h}}\right)^\top P_i(L) \left(\prod_{h=1}^{L-k} A_{i_{L-h}}\right) + Y_i(k) \tag{13}$$

where

$$Y_i(k) \triangleq \sum_{s=1}^{L-k-1} \left(\prod_{h=s}^{L-k-1} A_{i_{L-h}}\right)^\top X_i(L-s) \times \left(\prod_{h=s}^{L-k-1} A_{i_{L-h}}\right) + X_i(k).$$

Obviously, $P_i(k)$ in (13) satisfies $A_{i_k}^\top P_i(k+1) A_{i_k} - P_i(k) = -X_i(k)$. Thus, (10) holds due to $X_i(k) \succ 0$, $k \in \mathcal{I}[0,L-1]$, $i \in \mathscr{A}(L,\tau)$. Then, letting $k=0$ in (13), we arrive at

$$P_i(0) = \left(\prod_{h=1}^{L} A_{i_{L-h}}\right)^\top P_i(L) \left(\prod_{h=1}^{L} A_{i_{L-h}}\right) + Y_i(0). \tag{14}$$

First, since $P_i(L) \in \mathbb{S}^n_{\succ 0}$ and $X_i(k) \in \mathbb{S}^n_{\succ 0}$, it is easy to see that $P_i(k) \in \mathbb{S}^n_{\succ 0}$. Then, we consider $P_i(0) - P_j(L)$ in (11), which, using (14), can be rewritten as follows:

$$P_i(0) - P_j(L)$$
$$= \left(\prod_{h=1}^{L} A_{i_{L-h}}\right)^\top P_i(L) \left(\prod_{h=1}^{L} A_{i_{L-h}}\right) + Y_i(0) - P_j(L). \tag{15}$$

Given any admissible switching path $i \in \mathscr{A}(L,\tau)$, using the definition of state transition matrix $x(L) = \Phi(L,0)x(0)$, where $\Phi(L,0) = \prod_{h=1}^{L} A_{i_{L-h}}$, where the sequence $\{i_0,\ldots,i_{L-1}\}$ is admissible with respect to dwell-time τ, it can be obtained that $P_i(0) - P_j(L) = \Phi^\top(L,0)P_i(L)\Phi(L,0) + Y_i(0) - P_j(L)$.

If system (1) is GUAS, there exists a class $\mathcal{KL}$ function β such that $\|x(L)\| \leq \beta(\|x(0)\|, L)$ holds, which means $\|\Phi(L,0)x(0)\| \leq \beta(\|x(0)\|, L)$. Since β is a class $\mathcal{KL}$ function, it implies that $\lim_{L\to\infty} \beta(\|x(0)\|, L) = 0$, and as a result, one has $\lim_{L\to\infty} \|\Phi(L,0)x(0)\| = 0$, leading to $\lim_{L\to\infty} \Phi(L,0) = 0$. Thus, for any arbitrarily chosen $P_i(L) \in \mathbb{S}^n_{\succ 0}$, there exists an $\epsilon > 0$ such that

$$\lim_{L\to\infty} \Phi^\top(L,0)P_i(L)\Phi(L,0) - P_j(L) = -P_j(L) \prec -\epsilon I.$$

It implies that there exists a sufficiently large L^* such that, for any $L \geq L^*$, the following inequality holds:

$$\Phi^\top(L,0)P_i(L)\Phi(L,0) - P_j(L) \prec -\epsilon I. \tag{16}$$

Thus, for any $L \geq L^*$, it has $P_i(0) - P_j(L) \prec -\epsilon I + Y_i(0)$.

Moreover, since $X_i(k)$, $k \in \mathcal{I}[0,L-1]$, $i \in \mathscr{A}(L,\tau)$ are arbitrarily chosen, $X_i(k)$ can be adjusted to make $Y_i(0)$ sufficiently small so as to attain $P_i(0) - P_j(L) \prec 0$. Therefore, (11) holds.

(*b*) $\Rightarrow$ (*a*): Due to $P_i(k) \in \mathbb{S}^n_{\succ 0}$, for each interval $[nL,(n+1)L]$, we construct Lyapunov functions in the following form:

$$V_i(x(k)) = \begin{cases} x^\top(k)P_i(\theta(k))x(k), & k \in [nL,(n+1)L-1] \\ x^\top(k)P_i(L)x(k), & k = (n+1)L \end{cases} \tag{17}$$

where $i \in \mathscr{A}[L,\tau]$ and $\theta(k)$ is the artificial timer defined by (9). Then, for any admissible switching path $i \in \mathscr{A}[L,\tau]$, we have

$$\Delta V_i(x(k)) = V_i(x(k+1)) - V_i(x(k))$$
$$= x^\top(k)[A_{i_k}^\top P_i(\theta(k+1))A_{i_k} - P_i(\theta(k))]x(k)$$

and (10) ensures that

$$\Delta V_i(x(k)) < 0,\ k \in [nL,(n+1)L-1],\ n \in \mathbb{N}. \tag{18}$$

Furthermore, let us consider the concatenation between two switching paths. Suppose $j \in \mathscr{A}(L,\tau)$ and $i \in \mathscr{P}\mathscr{A}(j)$. Then, (11) implies that

$$V_i(x(k)) < V_j(x(k)),\ k = (n+1)L,\ n \in \mathbb{N}. \tag{19}$$

From (18) and (19), the value of Lyapunov function (17) always decreases with time. Therefore, the GUAS of system (1) can be established by the standard Lyapunov theorem [37].

(*b*) $\Rightarrow$ (*c*): Since (11) holds, using (15) implies that the following inequality can be obtained:

$$\left(\prod_{h=1}^{L} A_{i_{L-h}}\right)^\top P_i(L) \left(\prod_{h=1}^{L} A_{i_{L-h}}\right) - P_j(L) \prec -Y_i(0).$$

TABLE I
COMPUTATIONAL COMPLEXITY OF STATEMENTS (*a*) AND (*b*)
($M = |\mathscr{A}[L,\tau]|, m(i) = |\mathscr{P}\mathscr{A}(i)|, i \in \mathscr{A}[L,\tau]$)

Statement	Number of variables	Size of LMIs
(*b*)	$\frac{n(n+1)(L+1)M}{2}$	$nM(L-1) + n\sum_{i\in\mathcal{I}[1,M]} m(i)$
(*c*)	$\frac{n(n+1)M}{2}$	$n\sum_{i\in\mathcal{I}[1,M]} m(i)$

Moreover, by $Y_i(k)$ defined in (13), it can be obtained that $Y_i(0) \succ 0$; thus, we have

$$\left(\prod_{h=1}^{L} A_{i_{L-h}}\right)^\top P_i(L)\left(\prod_{h=1}^{L} A_{i_{L-h}}\right) - P_j(L) \prec 0$$

which means (12) holds, just by letting $P_i = P_i(L)$.

(*c*) ⇒ (*b*): Equation (10) and $P_i(k) \in \mathbb{S}^n_{\succ 0}$ have been established at the beginning of (*a*) ⇒ (*b*); hence, in the rest, we only need to consider (11). Since (12) holds, it implies that there exists an $\epsilon > 0$ such that

$$\left(\prod_{h=1}^{L} A_{i_{L-h}}\right)^\top P_i\left(\prod_{h=1}^{L} A_{i_{L-h}}\right) - P_j \prec -\epsilon I.$$

Again, letting $P_i(L) = P_i$ and using (15), we can derive the following inequality $P_i(0) - P_j(L) \prec -\epsilon I + Y_i(0)$. Since $X_i(k)$, $k \in \mathcal{I}[0, L-1]$, $i \in \mathscr{A}(L,\tau)$ are arbitrarily chosen, $X_i(k)$ can be adjusted to make $Y_i(0)$ sufficiently small so as to achieve $P_i(0) - P_j(L) \prec 0$. Therefore, (11) holds. ■

Remark 1: Theorem 3 is the main result in this note. Some observations are made as follows.

1) Statement (*b*) generalizes the lifted convex idea of Theorem 2 on the basis of a virtual clock. Unlike in [20] where the lifted idea was implemented over the dwell-time interval, Statement (*b*) is derived based on the cycle of a virtual clock. Moreover, it should be stressed that this generalization will finally achieve a nonconservative stability criterion, as long as the system is equipped with a virtual clock with a sufficiently long cycle.
2) Statement (*c*) can be viewed as an improvement of Theorem 1 proposed in [7] since the condition in Statement (*c*) is a necessary and sufficient condition in contrast to Theorem 1, which is only a sufficient one. However, as in Theorem 1, system matrices A_i are not convex in (12) due to the presence of intricate multiplication of A_i, that is, $\prod_{h=1}^{L} A_{i_{L-h}}$.
3) Similar to the relationship between Theorems 1 and 2, Statement (*b*) is an equivalent convexification of Statement (*c*). Despite the equivalence of two conditions, the computational complexities are different. The computational complexities are listed in Table I. In Table I, it is shown that Statement (*b*) has a higher computational complexity than Statement (*c*) for checking the stability of system (1). The numbers of LMIs and variables for Statement (*b*) relate to L, while Statement (*b*) does not. This is the cost of convexification while still maintaining the same conservativeness in stability analysis. Furthermore, the results in [33] (Theorems 3.2 and 3.5 in [33]) and the Statement (*b*) in Theorem 3 both employ the lifting idea, and therefore, they result in LMIs with similar representations, though the difference is that the work presented in [33] is developed based on a switching graph and our approach is developed for switching rules with dwell-time constraint. The computational complexities of Statement (*b*) and results in [33] should be at the same level, as given in Table I.

Remark 2: For minimum dwell-time computation, we can check the admissible dwell time for a switched system via checking the feasibility of LMIs in Theorem 3 starting from a certain L; however, in order to avoid large computational cost, we usually start from a small L. If the dwell time τ is known, the L can be selected starting from $L = \tau$. As in the case with unknown dwell time, we can start our approach from any arbitrary L. In practice, in order to avoid large computational cost, we usually start from a small L as we will show in an example.

TABLE II
COMPUTATION ON MINIMAL DWELL TIME $\tau_{\min}$

Methods	$\tau_{\min}$	Computational Time
(*b*) in Corollary 1 ($L = 3$)	1	0.101412 seconds
(*c*) in Corollary 1 ($L = 3$)	1	0.04883 seconds

TABLE III
ℓ_2-INDUCED GAIN COMPUTATION AND COMPUTATIONAL TIME (S)

	$L = 3$	$L = 4$	$L = 5$
ℓ_2-induced gain γ^*	7.7453	3.9781	3.9669
Computational time	0.376973	3.9781	16.938713

If the special case $\tau = 1$, that is, the arbitrary switching case, is taken into account, then it leads to $|\mathscr{A}(L,\tau)| = N^L$ so that the following corollary can be derived, which is the same as the result in [38].

Corollary 1: Consider switched system (1) under arbitrary switching, the following three statements are equivalent.

a) Switched system (1) is GUAS.

b) There exist a scalar $L \geq \tau$ and symmetric matrix sequences $P_i : \mathcal{I}[0, L] \to \mathbb{S}^n_{\succ 0}$, $i \in \mathcal{I}[1, N^L]$ such that

$$A_{i_k}^\top P_i(k+1)A_{i_k} - P_i(k) \prec 0,\ i_k \in \mathcal{I}[1, N] \tag{20}$$

$$P_i(0) - P_j(L) \prec 0 \tag{21}$$

hold for all $i, j \in \mathcal{I}[1, N^L]$ and $k = 0, \ldots, L-1$.

c) There exist a scalar $L \geq \tau$ and symmetric matrices $P_i \in \mathbb{S}^n_{\succ 0}$, $i \in \mathcal{I}[1, N^L]$ such that

$$\left(\prod_{h=1}^{L} A_{i_{L-h}}\right)^\top P_i\left(\prod_{h=1}^{L} A_{i_{L-h}}\right) - P_j \prec 0 \tag{22}$$

hold for all $i, j \in \mathcal{I}[1, N^L]$.

Proof: In arbitrary switching, the admissible cycle allows all possible switching paths; thus, it has $\mathscr{A}(L,\tau) = \{1, 2, \ldots, N^L\}$ and $\mathscr{P}\mathscr{A}(i) = \{1, 2, \ldots, N^L\}$ $\forall i \in \mathscr{A}(L,\tau)$. Thus, it can be proved based on Theorem 3 and is omitted. ■

Example 2: Let us consider system (1) with two subsystems as follows:

$$A_1 = \begin{bmatrix} 0.969 & 0.0761 \\ -0.7607 & 0.8929 \end{bmatrix},\ A_2 = \begin{bmatrix} 0.9997 & 0.0685 \\ -0.0068 & 0.7259 \end{bmatrix}.$$

In order to show the improvement made by our approach, we first use the approaches presented in [7] and [20], namely, Theorems 1 and 2, to compute the minimum dwell time $\tau_{\min} = 3$, and meanwhile, we apply the virtual clock approach. The computation results are presented in Table II.

It can be found that less conservative results can be obtained when we let $L = 3$, that is, $\tau_{\min} = 1$. Moreover, note that $\tau_{\min} = 1$ indicates that the system is GUAS under arbitrary switching, and this is explicitly a nonconservative result for the proposed switched system. The convergent state evolutions are shown by the simulation results in Fig. 1, where 1000 state trajectories are randomly generated. All these state trajectories converge to the origin to show the GUAS of the system. Furthermore, assuming that the initial state is $x(0) = [1\,1]^\top$,

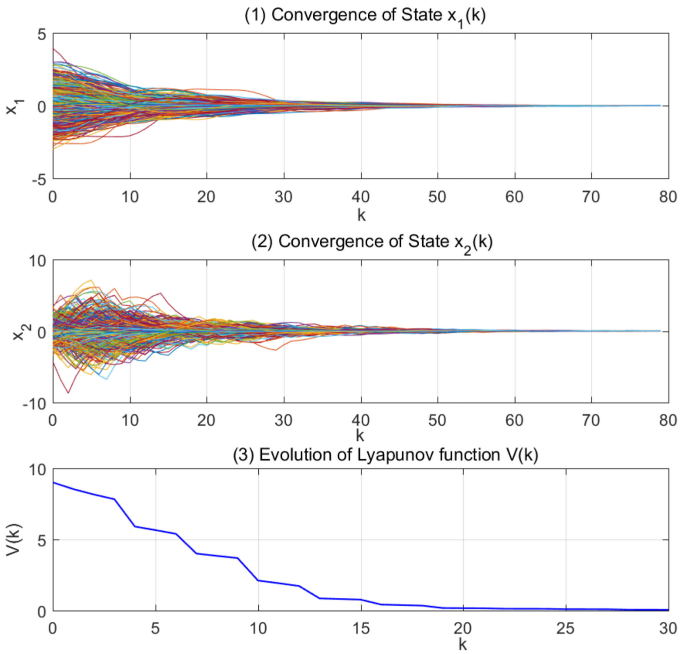

Fig. 1. Thousand randomly generated state trajectories and evolution of Lyapunov function. (a) Convergence of state $x_1(k)$. (b) Convergence of state $x_2(k)$. (c) Evolution of Lyapunov function $V(k)$.

the Lyapunov function derived from Statement (*b*) strictly decreases, as shown in Fig. 1, which guarantees the GUAS.

Though Statements (*b*) and (*c*) can both achieve the same result, the computational complexities are different as Table I indicates. The computational time is also given in Table II; it shows that Statement (*b*) needs to afford a higher computational cost, which is consistent with Table I.

According to Table I, the number of variables and LMIs will grow to a large number as L is a large number; this may lead to difficulties in the practical use of the developed approach. However, if the information of the switching rule is known, the computation cost will be affordable, even when L is large. Let us consider the example presented in [14]. The system matrices are given by

$$A_1 = \begin{bmatrix} 1 & 0.1 \\ -0.2 & 0.9 \end{bmatrix}, \; A_2 = \begin{bmatrix} 1 & 0.1 \\ -0.9 & 0.9 \end{bmatrix}.$$

The switching is a periodic one with $k_{\ell+1} - k_\ell = 15 \; \forall \ell \in \mathbb{N}$ and $\delta(0) = 1$. Thus, we can choose $L = 15$, and two admissible cycles $\mathcal{S}_1 = \{1, 1, \ldots, 1\}$ and $\mathcal{S}_2 = \{2, 2, \ldots, 2\}$ are sufficient to characterize the switching law and the corresponding virtual clock. Applying both Statements (*b*) and (*c*) in Theorem 3, the stability can be established with $\tau = 15$, which is consistent with the work presented in [14]. Moreover, it can be seen that $\tau = 15$ is a nonconservative result since the spectrum of $A_1^{14} A_2^{14}$ contains one eigenvalue outside the unit disc.

IV. ℓ_2-Induced Gain Computation

Involving control input $u(k) \in \mathbb{R}^m$, exogenous input disturbances $\omega(k) \in \mathbb{R}^l$, and output $y(k) \in \mathbb{R}^p$, we consider the following switched system in the rest of this paper:

$$\begin{aligned} x(k+1) &= A_{\sigma(k)} x(k) + B_{\sigma(k)} u(k) + E_{\sigma(k)} \omega(k) \\ y(k) &= C_{\sigma(k)} x(k) + D_{\sigma(k)} u(k) + F_{\sigma(k)} \omega(k) \end{aligned} \tag{23}$$

where B_i, C_i, D_i, E_i, and F_i are constant matrices with appropriate dimensions.

Definition 5: For $\gamma > 0$, system (23) with $u(k) = 0$ is said to be GUAS with an ℓ_2-gain performance at a level γ, if system (23) is GUAS when $u(k) = 0$ and $\omega(k) = 0$, and under zero initial conditions, the following inequality holds for all nonzero $\omega(k) \in \ell_2[0, \infty)$:

$$\sum_{k=0}^{\infty} y^\top(k) y(k) \le \gamma^2 \sum_{k=0}^{\infty} \omega^\top(k) \omega(k) \tag{24}$$

where γ is called the ℓ_2-gain, and the ℓ_2-induced gain of system (23) with $u(k) = 0$ is defined by $\gamma^* \triangleq \inf\{\gamma \ge 0 : (24) \text{ holds } \forall \omega(k) \in \ell_2[0, \infty), \, \omega(k) \neq 0\}$.

Theorem 4: Given a scalar $\gamma > 0$ and considering system (23) with $u(k) = 0$ and $\sigma(k) \in \mathscr{D}_\tau$, if there exist a scalar $L \ge \tau$ and symmetric matrix sequences $P_i : \mathcal{I}[0, L] \to \mathbb{S}^n_{\succ 0}$, $i \in \mathscr{A}(L, \tau)$ such that

$$\Theta_i(k) \prec 0, \; i \in \mathscr{A}(L, \tau), \; k = 0, \ldots, L-1 \tag{25}$$

$$P_i(0) - P_j(L) \prec 0, \; i \in \mathscr{P}\mathscr{A}(j), \; j \in \mathscr{A}(L, \tau) \tag{26}$$

where

$$\Theta_i(k) = \begin{bmatrix} -P_i(k) & * & * & * \\ 0 & -\gamma^2 I & * & * \\ P_i(k+1) A_{i_k} & P_i(k+1) E_{i_k} & -P_i(k+1) & * \\ C_{i_k} & F_{i_k} & 0 & -I \end{bmatrix}$$

and $i_k \in \mathcal{I}[1, N]$, then system (23) with $u(k) = 0$ and $\omega(k) = 0$ is GUAS and has an ℓ_2-gain γ.

Proof: The GUAS can be obtained from (25) and (26) by directly using (10) and (11) in Theorem 3. In the rest of the proof, we, therefore, only need to focus on proving the ℓ_2-gain performance.

Let

$$J = \sum_{k=0}^{\infty} \left(y^\top(k) y(k) - \gamma^2 \omega(k) \omega(k) \right) \tag{27}$$

and then, we construct a Lyapunov function $V_i(x(k))$ in the form of (17). Noting that $x_0 = 0$, J can be rewritten as follows:

$$\begin{aligned} J &= \sum_{h=0}^{\infty} \left(\sum_{k=hL}^{(h+1)L-1} \Gamma_i(k) + V_i(x(hL)) - V_i(x((h+1)L)) \right) \\ &= \sum_{h=0}^{\infty} \left(\sum_{k=hL}^{(h+1)L-1} \Gamma_i(k) \right) + \sum_{h=1}^{\infty} (V_i(x(hL)) - V_j(x(hL))) \end{aligned}$$

where $\Gamma_i(k) = y^\top(k) y(k) - \gamma^2 \omega^\top(k) \omega(k) + \Delta V_i(x(k))$. Using Schur complement formula, (25) yields

$$\Xi_i(k) = \begin{bmatrix} \Omega_i(k) & A_{i_k}^\top P_i(k+1) E_{i_k} + C_{i_k}^\top F_{i_k} \\ * & E_{i_k}^\top P_i(k+1) E_{i_k} + F_{i_k}^\top F_{i_k} - \gamma^2 I \end{bmatrix} \prec 0$$

where $\Omega_i(k) = A_{i_k}^\top P_i(k+1) A_{i_k} - P_i(k) + C_{i_k}^\top C_{i_k}$. Thus, it leads to $\Gamma_i(k) < 0$ since $\Gamma_i(k) = \xi^\top(k) \Xi_i(k) \xi(k)$, where $\xi(k) = [x^\top(k) \; \omega^\top(k)]^\top$. Moreover, (26) guarantees $V_i(x(hL)) - V_j(x(hL)) < 0$ $\forall h = 1, 2, \ldots$. Therefore, $J < 0$ can be established, which implies that the ℓ_2-gain performance can be established. The proof is complete. ■

Remark 3: As in stability analysis, a larger L would lead to a less conservative result at the expense of a higher computational complexity. Regarding ℓ_2-gain performance analysis, it means that a larger L will yield a smaller γ for the optimization problem $\min \; \gamma^2$ s.t. (25), (26) .

Example 3: Consider system (23) with two subsystems, and $u(k) = 0$, $A_i, i \in \{1, 2\}$ are the same as in Example 2; then, let $C_1 = C_2 = [0.1 \; 0.2]$, $E_1 = E_2 = [0.1 \; -0.1]^\top$, and $F_1 = F_2 = 0$.

In Example 2, the GUAS of the proposed system only can be established for $\tau_{\min} \geq 3$. As an extension of stability analysis, the ℓ_2-gain computation result in [20] is unable to compute the ℓ_2-gain for $\tau_{\min} < 3$. Using Theorem 4, we can compute the ℓ_2-induced gain for $\tau_{\min} < 3$. For example, with $\tau_{\min} = 1$, the ℓ_2-induced gain can be computed with $L \geq 3$. Table III presents the ℓ_2-induced gains with different L. It can be observed that the estimation of ℓ_2-induced gain decreases as L grows, but more computational time is required for the computation.

V. Virtual-Clock-Dependent $\mathcal{H}_\infty$ Control

Taking advantage of the convex feature in the virtual clock idea, a novel virtual-clock-dependent controller is introduced in the form of

$$u(k) = K_i(\theta(k))x(k),\ i \in \mathcal{I}[1, N] \tag{28}$$

where $\theta(k)$ is the artificial timer defined by (9). Substituting controller (28) into system (23), the closed-loop system is given by

$$\begin{aligned} x(k+1) &= \tilde{A}_{\sigma(k)}x(k) + E_{\sigma(k)}\omega(k) \\ y(k) &= \tilde{C}_{\sigma(k)}x(k) + F_{\sigma(k)}\omega(k) \end{aligned} \tag{29}$$

where $\tilde{A}_i = A_i + B_iK_i(\theta(k))$ and $\tilde{C}_i = C_i + D_iK_i(\theta(k))$. The design objective is to find proper feedback gains $K_i(\theta(k)),\ i \in \mathcal{I}[1, N]$ so as to ensure the ℓ_2-gain performance of system (29).

Theorem 5: Given a scalar $\gamma > 0$ and considering system (23) with $\sigma(k) \in \mathscr{D}_\tau$, if there exist a scalar $L \geq \tau$ and symmetric matrix sequences $Q_i : \mathcal{I}[0, L] \to \mathbb{S}^n_{\succ 0},\ i \in \mathscr{A}(L, \tau)$, matrix sequences $U_i : \mathcal{I}[0, L-1] \to \mathbb{R}^{m\times n},\ W_i : \mathcal{I}[0, L-1] \to \mathbb{R}^{n\times n},\ i \in \mathcal{I}[1, N]$ such that

$$\Xi_i(k) \prec 0,\ i \in \mathscr{A}(L, \tau),\ k = 0, \ldots, L-1 \tag{30}$$

$$Q_j(L) - Q_i(0) \prec 0,\ i \in \mathscr{P}\mathscr{A}(j),\ j \in \mathscr{A}(L, \tau) \tag{31}$$

where

$$\Xi_i(k) = \begin{bmatrix} \Xi_{i,1}(k) & * & * & * \\ 0 & -\gamma^2 I & * & * \\ \Xi_{i,2}(k) & E_{i_k} & -Q_i(k+1) & * \\ \Xi_{i,3}(k) & F_{i_k} & 0 & -I \end{bmatrix}$$

in which $\Xi_{i,1}(k) = Q_i(k) - W_i^\top(k) - W_i(k)$, $\Xi_{i,2}(k) = A_{i_k}W_i(k) + B_{i_k}U_i(k)$, and $\Xi_{i,3}(k) = C_{i_k}W_i(k) + D_{i_k}U_i(k)$, $i_k \in \mathcal{I}[1, N]$, then the closed-loop system (29) is GUAS with $\omega(k) = 0$ and has an ℓ_2-gain γ, where the virtual-clock-dependent feedback gains $K_i(\theta(k)),\ i \in \mathcal{I}[1, N]$ are given by

$$K_i(\theta(k)) = U_i(\theta(k))W_i^{-1}(\theta(k)),\ i \in \mathcal{I}[1, N] \tag{32}$$

and $\theta(k)$ is the timer defined by (9).

Proof: From feedback gains (32) and timer (9), we have

$$W_i(k) = K_i(k)U_i(k),\ k = 0, \ldots, L-1,\ i \in \mathcal{I}[1, N]. \tag{33}$$

Substituting (33) into (30), we arrive at

$$\begin{bmatrix} \Xi_{i,1}(k) & * & * & * \\ 0 & -\gamma^2 I & * & * \\ \tilde{A}_{i_k}W_i(k) & E_{i_k} & -Q_i(k+1) & * \\ \tilde{C}_{i_k}W_i(k) & F_{i_k} & 0 & -I \end{bmatrix} \prec 0. \tag{34}$$

In addition, from (30), we have

$$Q_i(k) - W_i^\top(k) - W_i(k) \prec 0 \tag{35}$$

which means that $W_i(k)$ is of full rank. Moreover, as $Q_i(k)$ is strictly positive definite, we also have

$$(Q_i(k) - W_i(k))^\top Q_i^{-1}(k)(Q_i(k) - W_i(k)) \succeq 0 \tag{36}$$

which is equivalent to

$$W_i^\top(k)Q_i^{-1}(k)W_i(k) \succeq W_i^\top(k). + W_i(k) - Q_i(k). \tag{37}$$

It follows that

$$\begin{bmatrix} -W_i^\top(k)Q_i^{-1}(k)W_i(k) & * & * & * \\ 0 & -\gamma^2 I & * & * \\ \tilde{A}_{i_k}W_i(k) & E_{i_k} & -Q_i(k+1) & * \\ \tilde{C}_{i_k}W_i(k) & F_{i_k} & 0 & -I \end{bmatrix} \prec 0$$

which is equal to

$$\begin{bmatrix} -Q_i^{-1}(k) & * & * & * \\ 0 & -\gamma^2 I & * & * \\ \tilde{A}_{i_k} & E_{i_k} & -Q_i(k+1) & * \\ \tilde{C}_{i_k} & F_{i_k} & 0 & -I \end{bmatrix} \prec 0. \tag{38}$$

Pre- and post-multiplying $\mathrm{diag}\{I, I, Q^{-1}(k+1), I\}$ and letting $P_i(k) = Q_i^{-1}(k)$, it can be equivalently expressed as follows:

$$\begin{bmatrix} -P_i(k) & * & * & * \\ 0 & -\gamma^2 I & * & * \\ P_i(k+1)\tilde{A}_{i_k} & P_i(k+1)E_{i_k} & -P_i(k+1) & * \\ \tilde{C}_{i_k} & F_{i_k} & 0 & -I \end{bmatrix} \prec 0.$$

Moreover, (31) leads to $P_i(0) - P_j(L) \prec 0,\ i \in \mathscr{P}\mathscr{A}(j),\ j \in \mathscr{A}(L, \tau)$. Therefore, by Theorem 4, the closed-loop system (29) is GUAS with $\omega(k) = 0$ and has an ℓ_2-gain γ. ■

Remark 4: In Theorem 5, the designed controller gains are both mode-dependent and time-dependent, that is, virtual-clock-dependent. The feedback gain at each instant is chosen not only based on the activated mode but also depending on the timer $\theta(k)$. For each subsystem, the controller gain actually has L values for the selection of different steps in an L-step sequence. Thus, in order to implement this virtual-clock-dependent controller, a virtual clock $\mathscr{C}_L$ needs to be equipped to monitor the value of timer $\theta(k)$ to select the proper gain.

Two special cases are considered in the sequel. First, if we only consider constant mode-dependent feedback gains for subsystems, the following corollary can be derived.

Corollary 2: Given a scalar $\gamma > 0$ and considering system (23) with $\sigma(k) \in \mathscr{D}_\tau$, if there exist a scalar $L \geq \tau$ and symmetric matrix sequences $Q_i : \mathcal{I}[0, L] \to \mathbb{S}^n_{\succ 0},\ i \in \mathscr{A}(L, \tau)$, matrices $U_i \in \mathbb{R}^{m\times n}$, $W_i \in \mathbb{R}^{n\times n},\ i \in \mathcal{I}[1, N]$ such that

$$\Xi_i(k) \prec 0,\ i \in \mathscr{A}(L, \tau),\ k = 0, \ldots, L-1 \tag{39}$$

$$Q_j(L) - Q_i(0) \prec 0,\ i \in \mathscr{P}\mathscr{A}(j),\ j \in \mathscr{A}(L, \tau) \tag{40}$$

where

$$\Xi_i(k) = \begin{bmatrix} Q_i(k) - W_i^\top - W_i & * & * & * \\ 0 & -\gamma^2 I & * & * \\ A_{i_k}W_i + B_{i_k}U_i & E_{i_k} & -Q_i(k+1) & * \\ C_{i_k}W_i + D_{i_k}U_i & F_{i_k} & 0 & -I \end{bmatrix}$$

then the closed-loop system (29) is GUAS with $\omega(k) = 0$ and has an ℓ_2-gain γ, where mode-dependent feedback gains $K_i,\ i \in \mathcal{I}[1, N]$ are

$$K_i = U_iW_i^{-1},\ i \in \mathcal{I}[1, N]. \tag{41}$$

Proof: Just let $W_i(k) = W_i$, $U_i(k) = U_i$, $\forall k \in \mathcal{I}[1, L]$, in Theorem 5, to complete the proof. ■

TABLE IV
ℓ_2-INDUCED GAINS OF THE CLOSED-LOOP SYSTEM (CG: COMMON GAIN METHOD; MD: MODE-DEPENDENT METHOD; VC: VIRTUAL-CLOCK METHOD)

	C.G.	M.D.	V.C. ($L = 2$)	V.C. ($L = 3$)	V.C. ($L = 4$)	V.C. ($L = 5$)
γ^*	3.2586	3.0047	3.0041	2.9936	2.9914	2.9908

Finally, if the switching signal cannot be detected online, which means $\sigma(k)$ is not available, then a common feedback control gain valid for all modes has to be designed.

Corollary 3: Given a scalar $\gamma > 0$ and consider system (23) with $\sigma(k) \in \mathscr{D}_\tau$, if there exist a scalar $L \geq \tau$ and symmetric matrix sequences $Q_i : \mathcal{I}[0, L] \to \mathbb{S}^n_{\succ 0}$, $i \in \mathscr{A}(L, \tau)$, matrices $U \in \mathbb{R}^{m\times n}$, $W \in \mathbb{R}^{n\times n}$ such that

$$\Xi_i(k) \prec 0,\ i \in \mathscr{A}(L, \tau),\ k = 0, \ldots, L - 1 \tag{42}$$

$$Q_j(L) - Q_i(0) \prec 0,\ i \in \mathscr{P}\mathscr{A}(j),\ j \in \mathscr{A}(L, \tau) \tag{43}$$

where

$$\Xi_i(k) = \begin{bmatrix} Q_i(k) - W^\top - W & * & * & * \\ 0 & -\gamma^2 I & * & * \\ A_{i_k} W + B_{i_k} U & E_{i_k} & -Q_i(k+1) & * \\ C_{i_k} W + D_{i_k} U & F_{i_k} & 0 & -I \end{bmatrix}$$

then the closed-loop system (29) is GUAS with $\omega(k) = 0$ and has an ℓ_2-gain γ, where the common feedback gain K is given by

$$K = UW^{-1}. \tag{44}$$

Proof: Just let $W = W_i$, $U = U_i$ $\forall i \in \mathcal{I}[1, N]$ in Corollary 2 to complete the proof. ■

Example 4: Consider the switched system (23) with the following system matrices:

$$A_1 = \begin{bmatrix} -0.92 & -0.34 \\ 1.0350 & -0.31 \end{bmatrix},\ A_2 = \begin{bmatrix} -0.92 & -0.34 \\ 1.0350 & -0.31 \end{bmatrix}$$

$$B_1 = \begin{bmatrix} -0.52 \\ 0.40 \end{bmatrix},\ B_2 = \begin{bmatrix} 0.85 \\ 0.05 \end{bmatrix},\ C_1^\top = \begin{bmatrix} -0.49 \\ 0.34 \end{bmatrix}$$

$$C_2 = \begin{bmatrix} 0.67 & -0.42 \end{bmatrix},\ D_1 = 1.44,\ D_2 = -0.36$$

$$E_1 = \begin{bmatrix} 0.90 \\ 0.97 \end{bmatrix},\ E_2 = \begin{bmatrix} 0.06 \\ -0.08 \end{bmatrix},\ F_1 = F_2 = 0.$$

The minimum dwell time is assumed to be $\tau_{\min} = 1$. In order to show the advantage of the virtual clock approach, Theorem 5 and Corollaries 2 and 3 are used to design virtual-clock-dependent, mode-dependent, and common feedback controllers, respectively, to minimize the ℓ_2-gain of the closed-loop system. The computation results are presented in Table IV. In Table IV, it can be seen that the virtual-clock-dependent controller has better performances than mode-dependent and common controllers since it yields smaller ℓ_2-induced gains for the closed-loop system. The common-gain controller has the largest and the most conservative design result; however, it does not need any virtual clock or mode activation detection, which is easy to be implemented in practice. Additionally, for the virtual-clock-dependent controller, a larger L can lead to a better $\mathcal{H}_\infty$ performance.

VI. CONCLUSION

Based on the novel virtual clock conception introduced in this note, two necessary and sufficient conditions for discrete-time switched linear systems under minimum dwell-time constraint are proposed. The lifted version is able to maintain the convex feature, which plays a crucial role for some further extensions. The nonconservativeness in stability analysis can be obtained as long as the length of the virtual clock is sufficiently long. Then, taking advantage of the convexity in the lifted condition, the extensions to ℓ_2-gain computation and $\mathcal{H}_\infty$ control problems are made. It shows that the virtual clock method outperforms the mode-dependent method and the common gain control method. Several numerical examples are given to illustrate the theoretical findings in this note. This note presents a framework of a virtual clock to improve the stability analysis for switched systems under dwell-time constraint. It should be mentioned that, according to Table I, the computational cost significantly increases as the number of modes and the length of the cycle of a virtual clock grow. Further studies are needed on how to reduce the computational complexity and make it applicable for a switched system with a large number of subsystems. Moreover, further investigations are also needed on how to select an appropriate L to avoid unnecessary computational costs.

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